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MAXIMUM PRINCIPLE IN THE PROBLEM OF TIME OPTIMAL RESPONSE WITH NONSMOOTH CONSTRAINTS

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The problem of optimal response [1, 2] with nonsmooth (generally speaking, nonfunctional) constraints imposed on the state variables is considered. This problem is used to illustrate the method of proving the necessary conditions of optimality in the problems of optimal control with phase constraints, based on constructive approximation of the initial problem with constraints by a sequence of problems of optimal control with constraint-free state variables. The variational analysis of the approximating problems is carried out by means of a purely algebraic method involving the formulas for the incremental growth of a functional [3, 4] and the theorems of separability of convex sets is not used.

Using a passage to the limit, the convergence of the approximating problems to the initial problem with constraints is proved, and for general assumptions the necessary conditions of optimality resembling the Pontriagin maximum principle [1] are derived for the generalized solutions of the initial problem. The conditions of transversality are expressed, in the case of nonsmooth (nonfunctional) constraints by a novel concept of a cone conjugate to an arbitrary closed set of a finite-dimensional space. The concept generalizes the usual notions of the normal and the normal cone for the cases of smooth and convex manifolds.

1. Statement of the problem. We consider a general problem of the time optimal response for systems of ordinary differential equations in the class of measurable controls $u(t)$ and absolutely continuous trajectories $x(t)$, $t_0 \leqslant t \leqslant t_1$

$$x^{\cdot} = f(x, u, t), \quad x = (x^1, \ldots, x^n)' \in \mathbf{R}^n \tag{1.1}$$

$$u(t) \in U(t) \subset \Omega, \quad t_0 \leqslant t \leqslant t_1 \tag{1.2}$$

$$(x(t_0),\ x(t_1)) \in G \subset \mathbf{R}^{2n} \tag{1.3}$$

$$I = t_1 - t_0 \to \inf \tag{1.4}$$

where the prime denotes transposition (x is a column vector), and the time t_0 is assumed fixed for the sake of simplicity. The constraints imposed on the state variables (1.3) are defined using an arbitrary closed set $G \subset \mathbf{R}^{2n}$ which may, in particular, have the form

$$\{z \in \mathbf{R}^{2n} : q_i(z) \leqslant 0,\ i = 1, \ldots, l;\ h_j(z) = 0,\ j = 1, \ldots, p\}$$

where $q_i(z)$ are arbitrary functions semi-continuous from below and $h_j(z)$ are arbitrary continuous functions.

In what follows, we shall assume that one of the projections of the set G on $\mathbf{R}^n$ is bounded and, that the following general conditions imposed on the parameters of the problem (1.1)—(1.4) all hold:

a) the space Ω is homeomorphic to a complete separable metric space, and the set $\Gamma_U = \{(u, t) \in \Omega \times [t_0, T]: u \in U(t)\}$ is analytic (mod 0) [5] for some T, $\infty > T \geqslant t_1$;

b) the functions $f(x, u, t)$ and $\partial f(x, u, t) / \partial x$ are continuous in x, B-measurable (mod 0) in (u, t) (this is fulfilled automatically if they are continuous in (x, u) and Lebesgue measurable in t) and satisfy the inequalities

$$\|f(x, u, t)\| \leqslant \mu(t)\, g(\|x\|), \quad \|\partial f(x, u, t) / \partial x\| \leqslant \mu_1(t) g_1(\|x\|)$$

where $g(s)$ and $g_1(s)$ are continuous on $[0, \infty)$, $\mu(t)$ and $\mu_1(t)$ are summable on $[t_0, T]$ and $g(s) = O(s)$ as $s \to \infty$;

c) the set $R(x, t) = \{r = (r_1, r_2) : r_1 = f(x, u, t), r_2 = \partial f(x, u, t) / \partial x, u \in U(t)\}$ is closed in the space $R^{n(n+1)}$.

The condition (b) guarantees the uniform boundedness of the set of admissible trajectories of the problem (1.1)—(1.4). Other general conditions of this type can be found in e.g. [6].

We shall call the generalized solution of the problem (1.1)—(1.4) the optimal set $\{x^\circ(t), \alpha_i^\circ(t), u_i^\circ(t),\ i = 1, \ldots, n+1\}$ in the extended [7] problem of minimizing the functional (1.4) with the constraints imposed on the state variables (1.3) given by the following relations:

$$x^{\cdot} = \sum_{i=1}^{n+1} \alpha_i f(x, u_i, t), \quad x = (x^1, \ldots, x^n)' \in \mathbf{R}^n \tag{1.5}$$

$$\alpha_i(t) \geqslant 0, \quad \sum_{i=1}^{n+1} \alpha_i(t) \equiv 1, \quad u_i(t) \in U(t), \quad t_0 \leqslant t \leqslant t_1,\ i = 1, \ldots, n+1 \tag{1.6}$$

2. The approximating problems. Let us denote by $\tau^\circ = t_1^\circ - t_0$ the minimum value of the functional (1.4) in the extended problem (1.3)—(1.6). This value is attained under the conditions (a)—(c), and coincides with the minimum value of the functional in the initial problem (1.1)—(1.4) for a wide class of problems correct in the extended form [5, 8]. Let us investigate an arbitrary numerical sequence $\{t_{1k}\}$, $k = 1, 2, \ldots$, satisfying the conditions $t_{1k} \to t_1^\circ$, $t_{1k} < t_1^\circ$, $k = 1, 2, \ldots$ Such a sequence can be effectively constructed using the discrete (finite-difference) approximations to the problems (1. 1) — (1. 4) correct in the extended form (the convergence of the discrete approximations to the functional is proved for similar problems in (*).

We introduce the following functional for each positive value of k :

$$I_k = \Phi(x(t_0),\ x(t_{1k})) = \rho(x(t_0),\ x(t_{1k});\ G) \to \inf, \qquad k = 1, 2, \ldots \quad (2.1)$$
$$\rho(x(t_0), x(t_{1k});\ G) = \min_{(y^\circ, y^1)} [(x(t_0) - y^\circ)'(x(t_0) - y^\circ) +$$
$$(x(t_{1k}) - y^1)'(x(t_{1k}) - y^1)]^{1/2}$$

where ρ is the distance of the boundary point $z_k = (x(t_0), x(t_{1k}))$ from the set G. Let us investigate the sequence of the approximating problems of minimizing the functional (2, 1) on the collection of sets $\{x(t),\ \alpha_i(t),\ u_i(t),\ i = 1, \ldots, n+1\}$, satisfying the relations (1. 5) and (1. 6) with $t_1 = t_{1k}$, $k = 1, 2, \ldots$. The approximating problem of optimal control (1. 5), (1. 6), (2. 1) represents, for every $k = 1, 2, \ldots$ a problem of minimizing a nonsmooth Mayer-type functional without additional constraints imposed on the state variables. When the conditions (a) — (c) hold, the problems (1. 5), (1. 6), (2. 1) always have solutions [8] in the form of the optimal sets $\{x_k^\circ(t),$ $\alpha_{ik}^\circ(t),\ u_{ik}^\circ(t),\ i = 1, \ldots, n+1\}$, $k = 1, 2, \ldots$. A theorem which follows, defines more exactly the character of the approximation of the initial problem with constraints imposed on the state variables by the problems (1. 5), (1. 6), (2. 1).

Theorem 2. 1. Let the assumptions (a) — (c) hold. Then the set of trajectories $\{x_k^\circ(t)\}$, $t_0 \leqslant t \leqslant t_{1k}$, $k = 1, 2, \ldots$ optimal in the problems (1. 5), (1. 6), (2.1) and continuously extended to the whole of the interval $[t_0, t_1^\circ]$, is relatively compact in the space $C[t_0, t_1^\circ]$ and its limit points represent the optimal trajectories of the widened problem (1. 3) — (1. 6). Moreover, a number $c > 0$ can be found such, that the following inequality holds for all $k = 1, 2, \ldots$:

$$\rho(x_k^\circ(t_0),\quad x_k^\circ(t_{1k});\ G) \leqslant c \int_{t_{1k}}^{t_1^\circ} \mu(t)\, dt \qquad (2.2)$$

Proof. Consider an arbitrary sequence of trajectories $\{x_k^\circ(t)\}$, $k = 1, 2, \ldots$, optimal in the problems (1. 5), (1. 6), (2. 1) and continuously extended to the interval $[t_0, t_1^\circ]$. By virtue of the condition (b) we can, following [9], find a number $r > 0$ such that

$$\|x_k^\circ(t)\| \leqslant r,\ \ t_0 \leqslant t \leqslant t_1^\circ,\ \ k = 1, 2, \ldots$$

Let us write the inequality

$$\|x_k^\circ(\tau_1) - x_k^\circ(\tau_2)\| \leqslant c \int_{\tau_1}^{\tau_2} \mu(t)\, dt \qquad (2.3)$$
$$\tau_1, \tau_2 \in [t_0, t_1^\circ], \quad \tau_1 \leqslant \tau_2, \quad k = 1, 2, \ldots, \quad c = \max_{\|x\| \leqslant r} g(\|x\|)$$

We see from it that the sequence $\{x_k^\circ(t)\}$, $t_0 \leqslant t \leqslant t_1^\circ$, $k = 1, 2, \ldots$ is continuous in the same degree. Using the Arzela-Ascoli theorem [10], we can separate from this sequence a subsequence, which converges uniformly on $[t_0, t_1^\circ]$ to some absolutely continuous (by virtue of (2. 3)) function $x^\circ(t)$, $t_0 \leqslant t \leqslant t_1^\circ$. Using the fact that the set of admissible velocities in the extended problem is convex and the theorem of measurable sampling [6, 8], we can find measurable functions $\alpha_i^\circ(t)$, $u_i^\circ(t)$, $t_0 \leqslant t \leqslant t_1^\circ$,

*) Mordukhovich, B. Sh. Convergence of the discrete approximations in the problems of optimal control. IV-th Mathematical Conference of the Belorussian Republic, Conf. Reports, pt. 1, Minsk, 1975.

$i = 1, 2, \ldots$ such that the set $\{x^\circ(t), \alpha_i^\circ(t), u_i^\circ(t), i = 1, \ldots, n+1\}$, $t_0 \leqslant t \leqslant t_1^\circ$ satisfies the relations (1.5) and (1.6). The proof of the theorem will now follow fully from the inequality (2.2). The validity of this inequality can be shown by remarking [8] that an optimal trajectory $x^*(t)$, $t_0 \leqslant t \leqslant t_1^\circ$ exists in the problem (1.3) — (1.6), for which $\rho(x^*(t_0), x^*(t_1^\circ); G) = 0$ and for which (2.3) also holds when $\tau_2 = t_1^\circ$, $\tau_1 = t_{1k}$, $k = 1, 2, \ldots$. Since the trajectories $x_k^\circ(t)$, $t_0 \leqslant t \leqslant t_{1k}$ are optimal, we can write in the problems (1.5), (1.6), (2.1), $k = 1, 2, \ldots$.

$$\rho(x_k^\circ(t_0), x_k^\circ(t_{1k}); G) \leqslant \rho(x^*(t_0), x^*(t_{1k}): G) \leqslant \| x^*(t_{1k}) - x^*(t_1^\circ) \|$$

The required inequality follows from the above expression. Theorem 2.1 is proved.

From the above theorem it follows that the process of constructing and solving the approximating problems (1.5), (1.6), (2.1) can be regarded as a constructive algorithm for obtaining an approximate solution of the extended problem (and of the initial problem, provided that the extended problem is still correct) with arbitrary constraints of the type (1.3). The inequality (2.2) characterizes in this case the degree of approximation of the constraints of the type (1.3), depending on the rate of convergence of $t_{1k} \to t_1^\circ$.

3. The maximum principle in the approximating problems. We derive the necessary conditions of optimality of the first order in the approximating problems (1.5), (1.6), (2.1) using the algebraic constructions of the method of incremental growth of the functional [3, 4] and the results of the theorem of multivalued measurable mappings [6, 8].

We denote by m_k the minimum value of the functional in the problem (1.5), (1.6), (2.1). This value is always positive by virtue of the choice of the sequence $\{t_{1k}\}$, $k = 1, 2, \ldots$. Let us introduce the Hamilton function $H(x, \psi, u, t) = \psi' f(x, u, t)$ for the system (1.1) and consider the following equation for the conjugate variables corresponding to (1.5):

$$\psi^{\cdot} = -\sum_{i=1}^{n+1} \alpha_i \, \partial H(x, \psi, u_i, t) / \partial x \tag{3.1}$$

Theorem 3.1. Let the conditions (a) and (b) hold. Then for every set $\{x_k^\circ(t), \alpha_{ik}^\circ(t), u_{ik}^\circ(t), i = 1, \ldots, n+1\}$, $t_0 \leqslant t \leqslant t_{1k}$ optimal in (1.5), (1.6), (2.1) and for almost every $t \in T_{ik} = \{t \in [t_0, t_{1k}], \alpha_{ik}^\circ(t) \neq 0\}$ the following maximum principle holds:

$$H(x_k^\circ(t), \psi_k^\circ(t), u_{ik}^\circ(t), t) = \sup_{u \in U(t)} H(x_k^\circ(t), \psi_k^\circ(t), u, t), \quad i = 1, \ldots, n+1 \tag{3.2}$$

where $\psi_k^\circ(t)$, $t_0 \leqslant t \leqslant t_{1k}$ is the corresponding absolutely continuous solution of the system (3.1) with the boundary conditions

$$\psi_k^\circ(t_0) = \frac{1}{m_k}(x_k^\circ(t_0) - y_k^\circ), \quad \psi_k^\circ(t_{1k}) = \frac{1}{m_k}(y_k^1 - x_k^\circ(t_{1k})) \tag{3.3}$$

where $y_k = (y_k^\circ, y_k^1)$ is a vector belonging to the set

$$M_k = \{y_k = (y_k^\circ, y_k^1) \in G : \rho(x_k^\circ(t_0), x_k^\circ(t_{1k}); y_k^\circ, y_k^1) = \rho(x_k^\circ(t_0), x_k^\circ(t_{1k}); G\}$$

Proof. The set $\{x_k^\circ(t), \alpha_{ik}^\circ(t), u_{ik}^\circ(t), i = 1, \ldots, n+1\}$, $t_0 \leqslant t \leqslant t_{1k}$ optimal in the problem (1.5), (1.6), (2.1) will be optimal in the problem of minimizing the Mayer-type functional

$$J_k = F(x(t_0),\ x(t_{1k})) = [(x(t_0) - y_k^\circ)'(x(t_0) - y_k^\circ) + (x(t_{1k}) - y_k^1)'(x(t_{1k}) - y_k^1)]^{1/2} \to \inf \tag{3.4}$$

with a smooth function $F(x^\circ, x^1)$, $(x^\circ, x^1) \notin G$ on the trajectories of the system (1.5), (1.6) under an arbitrary choice of the vector $y_k = (y_k^\circ,\ y_k^1) \in M_k$. Using the method of increments [3, 4], we can write the following formula for an increment of the functional in the problem (1.5), (1.6), (3.4):

$$\begin{aligned}\Delta J_k = {} & (\partial F(x_k^\circ(t_0))/\partial x^\circ - \psi_k^\circ(t_0))'\Delta x(t_0) - \\ & \int_{t_0}^{t_{1k}} \Big[\sum_{i=1}^{n+1} \alpha_i(t) H(x_k^\circ(t), \psi_k^\circ(t), u_i(t), t) - \\ & \sum_{i=1}^{n+1} \alpha_{ik}^\circ(t) H(x_k^\circ(t), \psi_k^\circ(t), u_{ik}^\circ(t), t)\Big] dt - \\ & \int_{t_0}^{t_{1k}} \Big[\sum_{i=1}^{n+1} \alpha_i(t)\partial H(x_k^\circ(t), \psi_k^\circ(t), u_i(t), t)/\partial x - \\ & \sum_{i=1}^{n+1} \alpha_{ik}^\circ(t)\partial H(x_k^\circ(t), \psi_k^\circ(t), u_{ik}^\circ(t), t)/\partial x]'\Delta x(t)\, dt - \\ & \int_{t_0}^{t_{1k}} o(\|\Delta x(t)\|)\, dt + o(\|\Delta x(t_{1k})\|) + o(\|\Delta x(t_0)\|)\end{aligned} \tag{3.5}$$

Here the set $\{x(t), \alpha_i(t), u_i(t), i = 1, \ldots, n+1\}$, $t_0 \leqslant t \leqslant t_{1k}$ satisfies the relations (1.5) and (1.6), $\Delta x(t) = x(t) - x_k^\circ(t)$ and the function $\psi_k^\circ(t)$ satisfies Eq. (3.1) with the right-hand side boundary conditions of (3.2) along the optimal set $\{x_k^\circ(t), \alpha_{ik}^\circ(t), u_{ik}^\circ(t), i = 1, \ldots, n+1\}$, $t_0 \leqslant t \leqslant t_{1k}$.

We shall prove that the left-hand side boundary condition of (3.3) also holds for the function $\psi_k^\circ(t)$. Assume the opposite, i.e.

$$\frac{1}{m_k}(x_k^\circ(t_0) - y_k^\circ) - \psi_k^\circ(t_0) = \partial F(x_k^\circ(t_0))/\partial x^\circ - \psi_k^\circ(t_0) = b \neq 0 \tag{3.6}$$

Consider the solution $x_k(t)$, $t_0 \leqslant t \leqslant t_{1k}$ of the system (1.5) with the initial condition $x_k(t_0) = x_k^\circ(t_0) + b\varepsilon$ corresponding to the controlling set $\{\alpha_{ik}^\circ(t), u_{ik}^\circ(t), i = 1, \ldots, n+1\}$. By virtue of the condition (b) and the Bellman-Gronwall lemma [4], the following inequality holds:

$$\|x_k(t) - x_k^\circ(t)\| \leqslant \varepsilon\|b\|\exp\Big(c_1\int_{t_0}^{t_{1k}} \mu_1(t)\, dt\Big), \quad t_0 \leqslant t \leqslant t_{1k}$$

$$c_1 = \max_{\|x\| \leqslant r} g_1\|x\|$$

From (3.5) we have

$$\Delta J_k = \varepsilon\|b\|^2 + o(\varepsilon) \tag{3.7}$$

Having chosen a sufficiently small ε in (3.7), we arrive at a contradiction with the optimality on the set $\{x_k^\circ(t), \alpha_{ik}^\circ(t), u_{ik}^\circ(t), i = 1, \ldots, n+1\}$, $t_0 \leqslant t \leqslant t_{1k}$ in the problem (1.5), (1.6), (3.4). Thus the assumption (3.6) is invalid and the function $\psi_k^\circ(t)$ satisfies the boundary conditions (3.3).

The following relation holds for the optimal set $\{x_k^\circ(t), \alpha_{ik}^\circ(t), u_{ik}^\circ(t), i = 1, \ldots, n+1\}$ and for almost every $t \in [t_0, t_{1k}]$:

$$\sum_{i=1}^{n+1} \alpha_{ik}^{\circ}(t) H(x_k^{\circ}(t), \psi_k^{\circ}(t), u_{ik}^{\circ}(t), t) = \tag{3.8}$$

$$\sup_{(\alpha_i, u_i)} \sum_{i=1}^{n+1} \alpha_i H(x_k^{\circ}(t), \psi_k^{\circ}(t), u_i, t)$$

$$\left(\sum_{i=1}^{n+1} \alpha_i = 1,\ \alpha_i \geqslant 0,\ u_i \in U(t),\ i = 1, \ldots, n+1\right)$$

where the operation sup is performed all (α_i, u_i) satisfying the conditions enclosed within the parentheses.

Assuming the opposite and using the theorems of measurable sampling, we find the measurable set $T \subset [t_0, t_{1k}]$, mes $T = \delta > 0$, and the measurable functions $\alpha_{ik}(t)$, $u_{ik}(t)$, $t \in T$, $i = 1, \ldots, n+1$, for which

$$\alpha_{ik}(t) > 0, \quad u_{ik}(t) \in U(t), \quad \sum_{i=1}^{n+1} \alpha_{ik}(t) \equiv 1, \quad t \in T$$

and the inequality

$$\sum_{i=1}^{n+1} \alpha_{ik}(t) H(x_k^{\circ}(t), \psi_k^{\circ}(t), u_{ik}(t), t) > \tag{3.9}$$

$$\sum_{i=1}^{n+1} \overset{\circ}{\alpha}_{ik}(t) H(x_k^{\circ}(t), \psi_k^{\circ}(t), \overset{\circ}{u}_{ik}(t), t), \quad t \in T$$

holds.

Let $\theta \in T$ belong to the set of points of approximative continuity [10] of the function

$$h_k(t) = \sum_{i=1}^{n+1} \alpha_{ik}(t) H(x_k^{\circ}(t), \psi_k^{\circ}(t), u_{ik}(t), t) -$$

$$\sum_{i=1}^{n+1} \alpha_{ik}^{\circ}(t) H(x_k^{\circ}(t), \psi_k^{\circ}(t), u_{ik}^{\circ}(t), t), \quad t \in T$$

which, by virtue of the Denjou theorem [10] has a complete measure on T. Consider a family of controls $\{\alpha_{ik}^{\varepsilon}(t), u_{ik}^{\varepsilon}(t), i = 1, \ldots, n+1\}$, $t_0 \leqslant t \leqslant t_{1k}$, $\varepsilon > 0$, admissible in the problem (1.5), (1.6), (3.4) and obtained by almost impulsive variation of the optimal control $\{\alpha_{ik}^{\circ}(t), u_{ik}^{\circ}(t), i = 1, \ldots, n+1\}$, $t_0 \leqslant t \leqslant t_{ik}$ in the interval $x_k^{\varepsilon}(t_0) = x_k^{\circ}(t_0)$

$$(\alpha_{ik}^{\varepsilon}(t), u_{ik}^{\varepsilon}(t))t = \begin{cases} (\alpha_{ik}(t))t\ u_{ik}(t), & t \in T_{\varepsilon} \\ (\alpha_{ik}^{\circ}(t), u_{ik}^{\circ}(t)), & t \in [t_0, t_{ik}] \setminus T_{\varepsilon} \end{cases} \quad i = 1, \ldots, n+1 \tag{3.10}$$

Let $T_{\varepsilon} = [\theta, \theta + \varepsilon) \cap T$ be a trajectory of the system (1.5) with the initial condition $x_k^{\varepsilon}(t)$, $t_0 \leqslant t \leqslant t_{1k}$ corresponding to the control (3.10). It can easily be shown that

$$\| x_k^{\varepsilon}(t) - x_k^{\circ}(t) \| \leqslant 2\varepsilon c \mu(t) \exp\left(c_1 \int_{t_0}^{t_{1k}} \mu_1(t)\, dt\right), \quad \theta \leqslant t \leqslant t_{1k} \tag{3.11}$$

By virtue of (3.11) and the choice of the point θ, the formula (3.5) defining the functional increments yields, for $\alpha_i(t) = \alpha_{ik}^{\varepsilon}(t)$, $u_i(t) = u_{ik}^{\varepsilon}(t)$, $i = 1, \ldots, n+1$, $t_0 \leqslant t \leqslant t_{1k}$, the following relation:

$$\Delta J_k = -\varepsilon \Big[\sum_{i=1}^{n+1} \alpha_{ik}(\theta) H(x_k^{\circ}(\theta), \psi_k^{\circ}(\theta),$$

$$u_{ik}(\theta), \theta) - \sum_{i=1}^{n+1} \alpha_{ik}{}^{\circ}(\theta) H(x_k{}^{\circ}(\theta), \psi_k{}^{\circ}(\theta), u_{ik}{}^{\circ}(\theta), \theta\Big] + o(\varepsilon)$$

which at fairly small ε contradicts the optimal set $\{x_k{}^{\circ}(t), a_{ik}{}^{\circ}(t), u_{ik}{}^{\circ}(t), i = 1, \ldots, n+1\}$, $t_0 \leqslant t \leqslant t_{1k}$ in the problem (1.5),(1.6),(3.4). Thus the inequality (3.9) is invalid, and this proves the validity of (3.8).

To complete the proof of the theorem, we must show that (3.8) implies that the condition of maximum (3.2) holds for almost all $t \in T_{ik} = \{t \in [t_0, t_{1k}] : \alpha_{ik}{}^{\circ}(t) \neq 0\}$.

We assume that (3.2) does not hold for some $t \in T_{ik}$, $i \in \{1, \ldots, n+1\}$. Then, using the conditions governing the weighing coefficients α_i, $i = 1, \ldots, n+1$, we can write

$$\sum_{i=1}^{n+1} \alpha_{ik}{}^{\circ}(t) H(x_k{}^{\circ}(t), \psi_k{}^{\circ}(t), u_{ik}{}^{\circ}(t), t) < \sup_{u \in U(t)} H(x_k{}^{\circ}(t), \psi_k{}^{\circ}(t), u, t) \leqslant$$
$$\sup_{(\alpha_i, u_i)} \sum_{i=1}^{n+1} \alpha_i H(x_k{}^{\circ}(t), \psi_k{}^{\circ}(t), u_i, t)$$

This contradicts (3.8) and thus completes the proof of Theorem 3.1.

4. Conjugate cones and generalized derivatives. In order to formulate the fundamental result, we introduce the concept of a cone $K_G(e)$ conjugated at the point $e \in G$ to an arbitrary nonempty closed set G of a finite-dimensional space. We also introduce the associated concept of a generalized derivative (D-derivative) $D\varphi(x)$ for an arbitrary function $\varphi(x)$ of a finite number of real variables, semi-continuous from below. We consider the following sets for any point s of a finite-dimensional space:

$$M(s) = \{z \in G : \rho(s, z) = \rho(s, G)\}, \quad P(s) = \{p : p = \gamma(s - z), z \in M(s), \gamma > 0\} \tag{4.1}$$

We shall call the cone $K_G(e)$ conjugated at the point $e \in G$ to the set G, a closed cone of the form

$$K_G(e) = \bigcap_{\delta > 0} \overline{\bigcup_{\|s-e\| \leqslant \delta} P(s)} \tag{4.2}$$

At the point $s = e$, the cone (4.2) is an envelope of the cone $P(s)$, semi-continuous from above in the Kuratovski [8] sense. It can be shown that for smooth and convex sets G the concept of a conjugate cone (4.2) can be reduced, respectively, to the usual concepts of a normal, and a normal cone in the sense of the convex analysis [11]. The normal cone becomes, in the Clarke's sense [12], the convex closure of the cone $K_G(e)$.

Using the concept of a conjugate cone, we shall now define the D-derivative for an arbitrary function $\varphi(x)$, $x \in \mathbf{R}^n$, semi-continuous from below and assuming values on the extended straight line $(-\infty, \infty]$ Let us denote by $E = \operatorname{epi}\varphi = \{(x, \mu) \in \mathbf{R}^{n+1} : \mu \geqslant \varphi(x)\}$ the supergraph of the function $\varphi(x)$ which is a closed set in $\mathbf{R}^{n+1}$.

We shall call the D-derivative $D\varphi(x)$ of the function $\varphi : \mathbf{R}^n \to (-\infty, \infty]$ at the point x, $\varphi(x) < \infty$ semi-continuous from below, a set of the form

$$D\varphi(x) = \{v \in \mathbf{R}^n : (v, -1) \in K_E(x, \varphi(x))\} \tag{4.3}$$

where $K_E(x, \varphi(x))$ is the cone (4.2) conjugated to the supergraph $E = \operatorname{epi}\varphi \subset \mathbf{R}^{n+1}$ at the point $(x, \varphi(x))$.

Thus the D-derivative of the function semi-continuous from below on $\mathbf{R}^n$, is a multivalued mapping of the space $\mathbf{R}^n$ onto the set of its closed subsets. In the case of smooth functions $\varphi(x)$, the set (4.3) consists of a single point and the meaning of the D-derivative becomes that of the classical analysis. If $\varphi(x)$ is convex on $\mathbf{R}^n$, then the D-derivative (4.3) coincides with the subdifferential $\partial\varphi(x)$ in the sense of the convex analysis [11]. The generalized Clarke's gradient [12] can be obtained using the construction (4.2) in which the conjugate cone $K_E(x, \varphi(x))$ is replaced by its convex closure, i.e. by the cone normal in the Clarke's sense.

We note that the sets $K_G(e)$ and $D\varphi(x)$ can become convex even in the simplest cases. For example, for $\varphi(x) = -|x|$, $x \in \mathbf{R}$, we have $D\varphi(0) = \{-1; 1\}$.

5. The fundamental result. We shall formulate and prove the fundamental result of this paper, namely, the maximum principle in the problem of the time optimal response with nonsmooth (nonfunctional) constraints.

Theorem 5.1. Let the assumptions (a)—(c) hold. Then the problem (1.1)—(1.4) has no generalized solution $\{x^\circ(t), \alpha_i^\circ(t), u_i^\circ(t), i = 1, \ldots, n+1\}$, $t_0 \leqslant t \leqslant t_1^\circ$, which would satisfy, for almost every $t \in [t_0, t_1^\circ]$ the maximum principle

$$H(x^\circ(t), \psi^\circ(t), u_i^\circ(t), t) = \sup_{u \in U(t)} H(x^\circ(t), \psi^\circ(t), u, t), \ i = 1, .., n+1 \quad (5.1)$$

where $\psi^\circ(t)$, $t_0 \leqslant t \leqslant t_1^\circ$ is the corresponding, absolutely continuous trajectory of the system (3.1) with the boundary conditions (transversality conditions)

$$(\psi^\circ(t_0), -\psi^\circ(t_1^\circ)) \in K_G(x^\circ(t_0), x^\circ(t_1^\circ)), \ \|\psi^\circ(t_0)\|^2 + \|\psi^\circ(t_1^\circ)\|^2 = 1 \quad (5.2)$$

Proof. Consider a sequence of trajectories $x_k^\circ(t)$, $t_0 \leqslant t \leqslant t_{1k}$ optimal in the approximating problems (1.5), (1.6), (2.1), and the corresponding sequence of conjugate trajectories $\psi_k^\circ(t)$, $t_0 \leqslant t \leqslant t_{1k}$ satisfying the conditions (3.1)—(3.3) $k = 1, 2, \ldots$ (Theorem 3.1 guarantees the existence of such functions). We shall assume that the functions $x_k^\circ(t)$, and $\psi_k^\circ(t)$, $k = 1, 2, \ldots$ are prolonged continuously to the whole interval $[t_0, t_1^\circ]$. From the condition (3.3) it follows that

$$\|\psi_k^\circ(t_0)\|^2 + \|\psi_k^\circ(t_{1k})\|^2 = 1, \quad k = 1, 2, \ldots \quad (5.3)$$

In analogy with the proof of Theorem 2.1, we can conclude that the sequence $\{\psi_k^\circ(t)\}$, $t_0 \leqslant t \leqslant t_1^\circ$ is relatively compact in the space $C[t_0, t_1^\circ]$. Let us isolate from the sequence $\{x_k^\circ(t), \psi_k^\circ(t)\}$, $t_0 \leqslant t \leqslant t_1^\circ$, $k = 1, 2, \ldots$, a uniformly convergent subsequence the limit $\{x^\circ(t), \psi^\circ(t)\}$ of which is a function absolutely continuous on $[t_0, t_1^\circ]$.

From (2.2), (3.3), (5.3) and the definition of the conjugate cone K_G it follows directly that the limit functions $x^\circ(t)$, $\psi^\circ(t)$, $t_0 \leqslant t \leqslant t_1^\circ$, satisfy the boundary conditions (1.3), (5.2). Using the convexity of the set

$$Q(x, \psi, t) = \Big\{(q_1, q_2) \in \mathbf{R}^{2n}: q_1 = \sum_{i=1}^{n+1} \alpha_i f(x, u_i, t),$$

$$q_2 = \sum_{i=1}^{n+1} \alpha_i \partial H(x, \psi, u_i, t) / \partial x, \ \alpha_i \geqslant 0, \ u_i \in U(t), \ \sum_{i=1}^{n+1} \alpha_i = 1\Big\}$$

and the conditions (a)—(c), we can show [8] that measurable functions $\alpha_i^\circ(t)$, $u_i^\circ(t)$, $i = 1, \ldots, n+1$, can be found which satisfy, together with the limits $x^\circ(t)$, $\psi^\circ(t)$, $t_0 \leqslant t \leqslant t_1^\circ$, the conditions (1.5), (1.6), (3.1).

Thus the set $\{x^\circ(t),\ \alpha_i^\circ(t),\ u_i^\circ(t),\ \ i = 1, \ldots, n+1\},\ t_0 \leqslant t \leqslant t_1^\circ$ is a generalized solution of the initial problem (1.1) — (1.4). To complete the proof we must show that the functions $x^\circ(t),\ \psi^\circ(t)\ u_i^\circ(t),\ i = 1, \ldots, n+1$ satisfy, for almost every $t \in [t_0, t_1]$, the maximum condition (5.1).

We introduce the notation

$$\lim_{k\to\infty} h_k^\circ(t) = h^\circ(t) = \sup_{(\alpha_i, u_i)} \sum_{i=1}^{n+1} \alpha_i H(x^\circ(t), \psi^\circ(t), u_i, t)$$

$$\sum_{i=1}^{n+1} \alpha_i = 1, \quad \alpha_i \geqslant 0, \quad u_i \in U(t), \quad i = 1, \ldots, n+1$$

assuming that the whole sequence $\{x_k^\circ(t), \psi_k^\circ(t)\},\ k = 1, 2, \ldots$ converges uniformly on $[t_0, t_1]$ to the limit $\{x^\circ(t),\ \psi^\circ(t)\}$. Using the relation (3.8), we can show that for almost every $t \in [t_0, t_1^\circ]$ we have

$$h_k^\circ(t) = \sum_{i=1}^{n+1} \alpha_{ik}^\circ(t) H(x_k^\circ(t), \psi_k^\circ(t), u_{ik}^\circ(t), t), \quad k = 1, 2, \ldots \tag{5.4}$$

$$h^\circ(t) = \sum_{i=1}^{n+1} \alpha_i^\circ(t) H(x^\circ(t), \psi^\circ(t), u_i^\circ(t), t)$$

From (5.4) we can conclude, in analogy with the proof of Theorem 3.1, that the control $\{\alpha_i^\circ(t), u_i^\circ(t), i = 1, \ldots, n+1\}$ satisfies the maximum condition (5.1) for almost every

$$t \in T_i = \{t \in [t_0, t_1^\circ]: \alpha_i^\circ(t) \neq 0\},\ i = 1, \ldots,\ n+1.$$

We remove the last constraint by considering the set

$$V(t) = \{v \in U(t) : H(x^\circ(t),\ \psi^\circ(t), v, t) = \sup_{u \in U(t)} H(x^\circ(t),\ \psi^\circ(t),\ u,\ t)\}$$

From the conditions (a) — (c) it follows that the set $V(t)$ is nonempty for almost every $t \in [t_0, t_1]$, and its graph $\Gamma_V = \{(u, t) \in \Omega \times [t_0, t_1^\circ],\ \ u \in V(t)\}$ is an analytic (mod 0) subset of the space $\Omega \times [t_0, t_1^\circ]$. Using the theorem of measurable sampling [6, 8] we find the measurable function $u^*(t) \in V(t),\ t_0 \leqslant t \leqslant t_1^\circ$ and use it to replace the controls $u_i^\circ(t)$ on the sets $[t_0, t_1^\circ] \setminus T_i,\ \ i = 1, \ldots, n+1$. The control set $\{\alpha_i^\circ(t),\ u_i^\circ(t), i = 1, \ldots, n+1\},\ \ t_0 \leqslant t \leqslant t_1^\circ$ modified in this manner, satisfies the maximum principle (5.1) for almost every $t \in [t_0, t_1^\circ]$ and generates the same trajectories $x^\circ(t)$ and $\psi^\circ(t)$ by virtue of the systems (1.5) and (3.1). This completes the proof of the theorem.

From Theorem 5.1 follows the maximum principle for the problem of the time optimal response with smooth convex constraints, and the result due to Clarke [13] for the similar problems of optimal control (with fixed time) in which the conjugated cone K_G is replaced, under the conditions of transversality (5.2), by its convex closure (normal cone in the Clarke's sense). The Clarke's method consists of reducing the initial problem (1.1) — (1.4) to the generalized biconvex Bolza problem which was investigated in [14] by the methods of convex analysis. Additional assumptions on the "calmness" of the initial problem arise in the course pursued by the above method, which are absent from the approximation method proposed in this paper.

The proof of Theorem 5.1 offers a glimpse into the possibility of obtaining the condition of the fundamental result (the condition of transversality (5.2)) on account of arbitrary nonempty contraction of the set $M(s)$ in the course of constructing the conjugate cone (4.1),(4.2).

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